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The Logic of Informal Proofs

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The Claims
As listed in the abstract I

- progress in the philosophy of mathematical practice requires a general positive account of informal proof (since almost all mathematical proofs are informal in the strictest sense, even if they are highly formalised);
- informal proofs are arguments that depend on their matter as well as their logical form (in other words, ‘informal’ is a poor English translation for *inhaltliche);*
- articulating the dependency of informal inferences on their content requires a reconception of logic as the general study of inferential actions (in informal proofs, content, or representations thereof, plays a role in inference as the object of such actions);

The Claims
As listed in the abstract II

- it is a decisive advantage of this conception of logic that it accommodates the many mathematical proofs that include actions on objects other than propositions;
- further, it explains the fact that mathematics is (aside from some elementary mental arithmetic and simple spatial arguments) essentially inscribed.
- this conception of logic facilitates an intimate connection between logical questions about rigour and the study of mathematical cultures and practices (since the logical constraints on inferential actions are enacted as cultural norms).

Outputs
Publications so far:

"How to think about informal proofs" 1-Jan-2011 *Synthese* p. 1-16.

"What Philosophy of Mathematical Practice Can Teach Argumentation Theory about Diagrams and Pictures" 2012 *The Argument of Mathematics* Aberdein, A. & Dove, I. (eds.). Springer p. 209-222.

One-part motivational argument
Against the Derivation Recipe model of proof

The view that a mathematical proof is a sketch of or recipe for a formal derivation requires the proof to function as an argument that there is a suitable derivation.

This is a mathematical conclusion, and to avoid a regress we require some other account of how the proof can establish it.

One-part motivational argument
Against the Derivation Recipe model of proof

Let P be a mathematician's proof for a theorem C. Then, on the Derivation Recipe model, P is not really a proof of C, but rather an argument to convince the reader that:

> C': there is a formal system S such that $\vdash_S \gamma$ where γ is the formula in S corresponding to C

So, proponents and opponents of the Derivation Recipe view agree that P is a compelling, rigorous argument (a proof) of a mathematical conclusion. They differ over whether P is a proof of C or C'.

Articulating the dependency of informal inferences on their content requires a reconception of logic as the general study of inferential actions (in informal proofs, content, or representations thereof, plays a role in inference as the object of such actions)

This is not so radical:

- Formal logic offers a huge range of systems
- Formal logic has been extended to all manner of matters (tense logic, deontic logic, modal logic, etc.)
- Consider arguments about moving furniture, or the possibility of a new gymnastic feat

Articulating the dependency of informal inferences on their content requires a reconception of logic as the general study of inferential actions (in informal proofs, content, or representations thereof, plays a role in inference as the object of such actions)

This is not so radical:

- Philosophy of experimental science—the experiment is no longer simply a source of protocol sentences. It is a locus of rational action.
- Mathematical proofs (or rather, their texts) are full of imperatives. The objects of these imperatives are often not propositions but rather mathematical items or representations thereof
- These actions are often only available in certain domains (co-set counting; divisibility arguments; ε-δ chasing; Euclidean diagram manipulation;…)

It explains the fact that mathematics is (aside from some elementary mental arithmetic and simple spatial arguments) essentially inscribed.

The real value of mathematical representations is not that they present information clearly (though they do) but that they offer themselves for manipulation.

Example from Polya:

Peirce: Icons

Define the real numbers $c_1, c_2, c_3, \dots c_n \dots$ by: $c_1c_2c_3\dots c_n = (n+1)^n$

Then trivially: $$\sum_1^\infty (a_1a_2\dots a_n)^{1/n} = \sum_1^\infty \frac{(a_1c_1a_2c_2a_3c_3\dots a_nc_n)^{1/n}}{n+1}$$

People who may possibly agree with me
I Irina Starikova: Cayley graphs of finitely generated groups

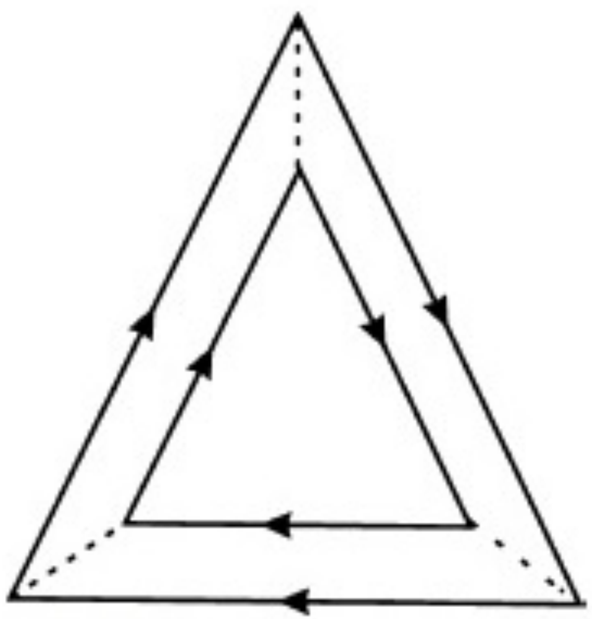


Fig. 4 A Cayley graph for the abelian group $C_3 \times C_2$

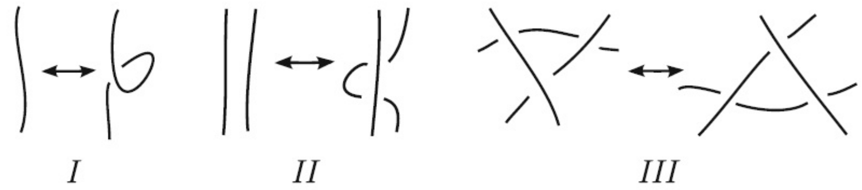
You can do things to Cayley graphs that you can't do to other representations of groups.

Starikova, Irina 2010 "Why do Mathematicians Need Different Ways of Presenting Mathematical Objects? The Case of Cayley Graphs" *Topoi* Volume 29, Issue 1 , pp 41-51

People who may possibly agree with me
II Silvia De Toffoli & Valeria Giardino

"…it will be shown that knot diagrams are *dynamic* by pointing at the moves which are commonly applied to them. For this reason, experts must develop a specific form of enhanced *manipulative* imagination, in order to draw inferences from knot diagrams by performing *epistemic* actions."

"Forms and Roles of Diagrams in Knot Theory"
Erkenntnis 79 (4):829-842 (2014)



The three Reidemeister moves

People who may possibly agree with me
III Dirk Schlimm & Andy Arana

As announced in Helsinki:

Geometric reasoning and geometric content

Dirk Schlimm

Philosophy, McGill University, Montreal, CANADA

My notes say, “Some mathematics is object-oriented“

They haven’t published yet…

People who may possibly agree with me
IV Andrei Rodin

In the same session in Helsinki:

Constructive Axiomatic Method in Euclid, Hilbert and Voevodsky

Andrei Rodin

Institute of Philosophy, Russian Academy of Sciences, Moscow, RUSSIAN FEDERATION

People who may possibly agree with me
IV Andrei Rodin

The received notion of axiomatic theory as a set of *propositions* (fully interpreted or not) provided with a relation of deducibility is not adequate to the successful practice of axiomatic thinking in mathematics and physics.

People who may possibly agree with me
IV Andrei Rodin

- The above claim equally concerns the old mathematics of Euclid's *Elements* and the very recent axiomatic *Homotopy Type theory* (HoTT).

People who may possibly agree with me
IV Andrei Rodin

- The key difference is this: a constructive theory, generally, is not a set of propositions; it treats propositions as certain types of objects along with certain other, non-propositional, types of objects. The formal distinction between propositional and non-propositional types will be explained in what follows.
- Warning: My use of the term "constructive" is not new and found, in particular in Hilbert&Bernays 1934 and Kolmogorov 1932. Nevertheless it significantly differs from other current uses of the same term.

People who may possibly agree with me
IV Andrei Rodin

Rules do <u>not</u> reduce to rules for handling propositions. Other types of theoretical objects should be equally handled according to certain rules.

I leave it open whether or not <u>only</u> rules for handling propositions qualify as *logical*. The standard axiomatic approach suggests the answer in positive. But one may wish to understand the scope of logic differently following Kant and some other influential philosophers.

People who may possibly agree with me
IV Andrei Rodin

- The constructive axiomatic method is evidently better implementable on **computer** than the received method. Its modern version has been implemented through COQ, AGDA and some other software.

Challenges
From people who definitely disagree with me

- Is it logic? 'logic' suggests systematisation, codification, which your view cannot supply. Also, logical principles or rules of inference should be fundamental. Elements of logic should be in some sense elementary. 'Divisibility' arguments are not elementary, because they can be further analysed and explained.
- A general account? Didn’t we PMP people decide that it’s all particular, contextual, situated, etc.?
- The last claim about cultures and practices is a dodgy promissory note…