A case of BHK decidability:
Dag Prawitz's proof- and ground-theoretic semantics

Antonio Piccolomini d'Aragona
antonio.piccolomini-daragona@etu.univ-amu.fr

Aix-Marseille University / CEPERC
"La Sapienza" University of Rome

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A first remark

Three reasons for the problem being a problem:

1. Proofs are intuitively taken to be recognisable as such.
2. Meaning is use, but use is something we are aware of.
3. Michael Dummett's manifestability requirement.

Overview of the talk

1. The BHK clauses for $\wedge$ and $\rightarrow$

BHK clause for $\wedge$

A proof of $\alpha \wedge \beta$ is an ordered pair

$$\langle \pi_\alpha, \pi_\beta \rangle$$

with π_α proof of α and π_β proof of β.

BHK clause for $\rightarrow$

A proof of $\alpha \rightarrow \beta$ is an effective function ϕ such that, for every π_α proof of α,

$$\phi(\pi_\alpha)$$

is a proof of β.

1. BHK functions

What is an effective function? As Rosza (Peter 1959) shows, it seems that, on pain of regressive explanations, we cannot provide an explicit definition. The notion is primitive.
No upper bound on the complexities of our functions. Explicit calculations could be longer than the expected length of the Universe.
Functions may involve free variables, which need to be substituted with proofs belonging to infinitary and non-regimented domains.
In short, one could be in possession of a BHK function that transforms proofs of α into proofs of β, but do not realize this fact; know *how* to get a proof of β out of a proof of α and not *that* this actually obtains.
Can we call a function of such a kind a proof? Can we be in possession of a proof without seeing that we are? It seems that we cannot.

1. Georg Kreisel's solution

BHKreisel clause for $\to$

A proof of $\alpha \to \beta$ is an order pair

$$\langle \phi, \tau \rangle$$

with ϕ effective function such that, for every π_α proof of α,

$$\phi(\pi_\alpha)$$

is a proof of β, and τ proof of this fact.

According to many authors (e.g. Dummett, Prawitz) this solution is wrong: it uses the very notion of proof that we aim to define.
[Although one may object that Kreisel puts on τ restrictions that could make the picture viable]

1. System recognition algorithms

Given a specific formal system S, whose set of derivations is Δ_S, there is available relative to them a recognition algorithm $\mathtt{Alg}_S$ such that, say,

$$\mathtt{Alg}_S(d) = \begin{cases} 1 & \text{if } d \in \Delta_S \\ 0 & \text{if } d \notin \Delta_S \end{cases}$$

Can we require that effective functions be decidable thanks to an algorithm of such a kind?
This would be to require that all the effective functions we need be generated by a decidable formal system. And because of Gödel's theorems, this cannot be already if simple Peano arithmetic is taken into account.
[Even for first-order logical provability (assuming that completeness holds - e.g. Prawitz's conjecture), the most one can obtain is *semi*-decidability]

1. Universal recognition algorithm - grounding trees

Let us consider upwards *grounding trees*.

Grounding tree for $\wedge$

1 $\langle \pi_\alpha, \pi_\beta \rangle$ BHK proof of $\alpha \wedge \beta$

2.1 $\pi_\alpha^* \overset{\texttt{norm}}{=} \pi_\alpha$ BHK proof of α

2.2 $\pi_\beta^* \overset{\texttt{norm}}{=} \pi_\beta$ BHK proof of β

... The tree terminates.

Grounding tree for $\rightarrow$

1 $\lambda x.b(x)$ is a BHK proof of $\alpha \rightarrow \beta$

2 a) b BHK proof of β under α, if α has no proof.
The tree terminates.

b) $\{b(\pi_\alpha^1)^* \overset{\texttt{norm}}{=} b(\pi_\alpha^1), ..., b(\pi_\alpha^n)^* \overset{\texttt{norm}}{=} b(\pi_\alpha^n), ...\}$ BHK proofs of β, if α has BHK proofs $\{\pi_\alpha^{1*} \overset{\texttt{norm}}{=} \pi_\alpha^1, ..., \pi_\alpha^{n*} \overset{\texttt{norm}}{=} \pi_\alpha^n, ...\}$.
... The tree terminates but may be infinite in width.

1. Universal recognition algorithm - negative result

Grounding trees are not recursively enumerable. Otherwise, we would have a uniform procedure that, when presented with $\lambda x.b(x)$ BHK proof of $\alpha \to \beta$, is able to choose between a) and b). So, we would have a uniform procedure to decide whether α is provable or not.

Suppose now that we have a universal recognition algorithm `Alg` such that

$$\texttt{Alg}(d, \alpha) = \begin{cases} 1 & \text{if } d \text{ is a BHK proof of } \alpha \\ 0 & \text{if } d \text{ is not a BHK proof of } \alpha \end{cases}$$

Let $\lambda x.b(x)$ be actually a BHK proof of $\alpha \to \beta$; then, it must be

$$\texttt{Alg}(b, \alpha \vdash \beta) = 1$$

and it is difficult to imagine an algorithm that behaves in this way without choosing between a) and b). But then, grounding trees would be recursively enumerable.

2.1 Argument structure

In Prawitz's proof-theoretic semantics (Prawitz 1973) one reasons with argument structures arranged in tree form, e.g.

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha \quad \beta \quad [\gamma\wedge\delta]}{\delta\to\alpha} \quad \delta\wedge\bot}{\beta\wedge(\alpha\to\bot)}}{\beta\to\bot} \qquad \dfrac{[\theta] \quad \gamma}{\delta\to(\gamma\to\beta)}\,\theta}{\bot\to((\alpha\to\beta)\to\bot)}\,\gamma\wedge\delta$$

An argument is a pair $\langle \Delta, F \rangle$, where Δ is an argument structure and F is a set of constructive functions associated to non-introductory rules in Δ, e.g.

$$\dfrac{\dfrac{\begin{matrix}\Delta_1\\ \alpha_1\end{matrix} \qquad \begin{matrix}\Delta_2\\ \alpha_2\end{matrix}}{\alpha_1\wedge\alpha_2}\,(\wedge_I)}{\alpha_i}\,(\wedge_{E,i}) \quad\Rightarrow\quad \begin{matrix}\Delta_i\\ \alpha_i\end{matrix} \qquad\qquad \dfrac{\dfrac{\begin{matrix}[\alpha]\\ \Delta_1\\ \beta\end{matrix}}{\alpha\to\beta}\,(\to_I) \qquad \begin{matrix}\Delta_2\\ \alpha\end{matrix}}{\beta}\,(\to_E) \quad\Rightarrow\quad \begin{matrix}\Delta_2\\ [\alpha]\\ \Delta_1\\ \beta\end{matrix}$$

2.1 Valid arguments

[This is a simplified version]
An argument $\langle \Delta, F \rangle$ is said to be valid iff:
(1) Δ is closed and ends with an introductory inference $\Rightarrow$ each of its immediate substructures is valid;
(2) Δ is closed and ends with a non introductory inference $\Rightarrow$ the reduction functions in F provide a method for reducing Δ to an argument structure Δ^* ending with an introductory inference and such that $\langle \Delta^*, F \rangle$ is valid;
(3) Δ is open with assumptions $\alpha_1, ..., \alpha_n \Rightarrow$ for every expansion F^+ of F, for every valid $\langle \Delta_i, F^+ \rangle$ for α_i, the result of replacing Δ_i for α_i in Δ gives rise to a valid argument.

2.1 Decidability problems in proof-theoretic semantics

The reduction of a valid closed non-canonical argument may be unfeasibly complex, and beyond our computational skills.
In order to see that an open argument is valid, one should perform infinitely many substitutions by taking into account non-regimentable expansions of a given set of reduction functions. That expansions are not regimentable implies that no decidable formal system can recursively generate all the needed compositions of reduction functions.
By contraposition. Suppose that S recursively generate all the compositions of reduction functions and let F be a set of reduction functions.
Step 1 S generates $f_1^1 \circ ... \circ f_h^1 \Rightarrow F^1 \stackrel{def}{=} F \cup \{f_1^1, ..., f_h^1\}$
...
Step n S generates $f_1^n \circ ... \circ f_k^n \Rightarrow F^n \stackrel{def}{=} F^{n-1} \cup \{f_1^n, ..., f_k^n\}$
...
Hence, it is not decidable whether $\langle \Delta, F \rangle$ is a valid argument for arbitrary α.

2.2 Basic question and grounds

The theory of grounds aims to explain why and how valid inferences have the epistemic power to confer evidence on their conclusion under assumed evidence for their premises. Premises and conclusion are taken to be judgements or assertions; they can be categorical [notation $\vdash \alpha$], hypothetical [notation $\alpha_1, ..., \alpha_n \vdash \beta$] or open [notation $\vdash \alpha(x_1, ..., x_n)$] (or both hypothetical and open).

Evidence is accounted for in terms of the key-notion of ground. Grounds are epistemic abstract entities. They are abstract because they are conceived of as what one must be in possession of in order to be justified in judging or asserting; they are epistemic because possession of grounds can be attained by performing knowledge operations of appropriate kind.

A basic tenet is the cartesian idea that proofs are chains of valid inferences. In the theory of grounds Prawitz reverses the inferences/proofs order of explanation of his old proof-theoretic semantics, so as to provide proofs with epistemic force and justify deductive compulsion.

2.2 Primitive language of grounds

Prawitz describes grounds as terms of an open typed λ-calculus. Following the formulas-as-types conception by Howard (Howard 1980), types come from formulas of a background first-order language L (with chosen range domain). One starts with we may call a *primitive* language G, whose alphabet is

- Ground-constants c^α for some atomic $\alpha \in L$
- Ground-variables ξ^α for $\alpha \in L$
- Primitive operations for logical constants $I_\wedge$ and $I_\rightarrow$

and whose set of typed terms $\mathbb{T}$ is

- $c^\alpha, \xi^\alpha : \alpha \in \mathbb{T}$
- $T : \alpha, U : \beta \in \mathbb{T} \Rightarrow I_\wedge(T, U) : \alpha \wedge \beta \in \mathbb{T}$
- $T : \beta \in \mathbb{T} \Rightarrow I_\rightarrow \xi^\alpha(T) : \alpha \rightarrow \beta \in \mathbb{T}$

2.2 Non primitive operations and clauses

We can already state a clause for atomic and conjunction grounding

(At) c^{α} ground for α

($\wedge$) T ground for α and U ground for β, $I_{\wedge}(T, U)$ ground for $\alpha \wedge \beta$ [pair-forming]

Grounds for implication forces us to consider *non*-primitive operations. An operation on grounds of type $\alpha_1, ..., \alpha_n \vdash \beta$ is unrestrictedly conceived of as an effective function f such that, if g_i is a ground for α_i, $f(g_1, ..., g_n)$ is a ground for β. Thus

($\rightarrow$) T operation on grounds of type $\alpha \vdash \beta$, $I_{\rightarrow}\xi^{\alpha}(T)$ ground for $\alpha \rightarrow \beta$ [λ-abstraction]

The problem is now how to express in general operation on grounds as terms of our λ-language. Remember that G is to be thought of as open.

2.2 Non-primitive languages

In order to bring non-primitive operations in our language of grounds, we have to consider *expansions* G^+ of G. An expansion is obtained by adding to G

- Non-primitive operations on grounds $F_1, ..., F_n$
- Defining equations ε_i for each added F_i, using already available tools

For example, we can introduce the non-primitive $E_{\wedge,1}$, $E_{\wedge,2}$ and $E_{\rightarrow}$ by defining them through $I_{\wedge}$ and $I_{\rightarrow}$:

$$E_{\wedge,i}(I_{\wedge}(T_1, T_2)) = T_i \text{ for } i = 1, 2 \text{ [left, right projection]}$$
$$E_{\rightarrow}(I_{\rightarrow}\xi^{\alpha}(T(\xi^{\alpha}), U) = T(U) \text{ [application]}$$

$E_{\wedge,1}$, $E_{\wedge,2}$ and $E_{\rightarrow}$ are provably grounds of type, respectively, α_1, α_2 and β when applied to grounds of type, respectively, $\alpha_1 \wedge \alpha_2$ and $\alpha \rightarrow \beta, \alpha$.

2.2 Ground-theoretic (valid) inferences

What kind of inferences have we to take into account so as to answer the main ground-theoretic question about the compelling power of valid inferences? Prawitz maintains, reasonably, that inferences cannot simply be transitions: in passing from premises to conclusion, we do something more than just moving downwards. If they just were transitions that the agent announces, why should the agent be compelled? After all, he is now just saying "this after those", presenting the result as holding because of the hypotheses: can this be sufficient to make him justified at all?

Inferences are aware applications of operations on the grounds one considers oneself to have for the premises with the aim to obtain a ground for the conclusion. If the inference is valid, the applied operation actually produces a ground for the conclusion; as the agent is aware of applying such an operation, he has a kind of epistemic insight on a ground growing up in his mind, and hence, of beeing entering a mental state of justification.

2.2 Decidability problems in the theory of grounds

Suppose that a closed term T is a ground, but contains some non-primitive operations. In order to see that T is a ground we have to reduce it to normal form, and this could again be unfeasibly complex.

Suppose that an open term T is an operation on grounds, but contains non-primitive operations. In order to see that T is an operation on grounds, we have to perform infinitely many substitutions from non-regimentable expansions of G.

Since non-primitive operations F are fixed by equations of the kind

$$F(T_1, ..., T_n) = U$$

the problem reduces to decide whether F is well-defined with respect to a desired operational type $\alpha_1, ..., \alpha_n \vdash \beta$. And it is clear that this cannot be the case. So, it is again not decidable whether T denotes a ground for $\alpha_1, ..., \alpha_n \vdash \beta$ (possibly, for $n = 0$).

3.1 Looking in states of mind

In Prawitz's ground-theoretic approach, there is paid more attention to what happens in one's mind when inferring or proving.

In proof-theoretic semantics, when one is in possession of a valid argument, one has performed certain validity-preserving transitions. If the transitions are meaning-constitutive, one is in possession of a canonical valid argument, whose validity is immediately recognizable; but if the steps are not meaning-constitutive, a non-canonical valid argument is got, and to recognize validity one has to reason on what one has made.

Instead, the terms of the theory of grounds are used to describe grounds; when one performs an inference step, one does not obtain a term, but a ground, and terms only serve to make one's mental deductive activity explicit.

3.2 Acts and objects

In addition, the theory of grounds is characterized by a strong operational view on inferences and, above all, on proofs. Inferences and proofs are acts, not objects.
In proof-theoretic semantics, valids arguments are both objects of evidence, and acts devoted to the production of such objects. This implies that the decidability issue concerns the very way in which evidence is accounted for; although decidability only becomes troubling when possession of evidence is taken into account, we cannot split our being, or gettin in possession of evidence, and what we are in possession of when we are justified.
Instead, in the theory of grounds we have objects on one side - i.e. (terms for) grounds - and, completely kept apart, acts that produce these objects - i.e. inferences and proofs. And when we perform these acts, we are aware of applying operation that, as said above, produce grounds. So, the operational view provides inferences and proofs with a higher epistemic import than that of proof-theoretic semantics.

4.1 How to understand decidability

For what said above, decidability cannot be looked at in an algorithmic way. Perhaps, one might more properly speak of "recognizability" or, staying closer to Kreisel, of "informal decidability". But what is such a phenomenon expected to be?

At least two generality degrees:

(GD) general decidability: we have a unique, homogeneus, non-algorithmic procedure allowing us to decide whether x is a BHK proof, a valid argument or a ground for y.

(LD) local decidability: we have a collection of reasonings such that, whenever x is presented to us, we can reason on how x is made and decide whether it is a BHK proof, a valid argument or a ground for y.

4.2 Two decidability claims

Depending on our choice, we will have two claims, one stronger and one weaker, to re-formulate the idea that provability is decidable - the stronger from (GD), and the weaker from (LD):

(S) there is a procedure $\mathtt{P}$ such that, for every x and y, $\mathtt{P}(x, y)$ informs us whether x is a BHK proof, a valid argument or a ground for y.

(W) for every x and y, there is a reasoning $\mathtt{P}$ such that $\mathtt{P}(x, y)$ informs us whether x is a BHK proof, a valid argument or a ground for y.

(W) seems reasonable: BHK proofs, valid arguments and grounds are epistemic objects, whose property should be looked upon as always in principle knowable. Moreover, if *reductio ad absurdum* holds, a denial of (W) would imply the existence of an epistemic object with an absolutely unknowable property. (S) is instead highly implausible. Even if decidability is not taken in an algorithmic sense, it is not clear which the procedure it foreshadows is, and it is not at all obvious how - and that - such a procedure can be found.

Essential bibliography

Howard W (1980) The formula-as-types notion of construction. In: Hindley JR and Seldin JP (eds) To H. B. Curry: Essays on Combinatoriy Logic, Lambda Calculus and Formalism. Academic Press, London.

Martin-Löf P (1984) Intuitionistic type theory. Bibliopolis, Napoli.

Prawitz D (1973) Towards a foundation of a general proof-theory. In: Suppes P (ed) Logic methodology and philosophy of science IV. North-Holland Publishing Company, Amsterdam, p 225 - 307.

Prawitz D (2015) Explaining deductive inference. In: Wansing H (ed) Dag Prawitz on proofs and meaning. Springer, Berlin Heidelberg New York.

Peter R (1959) Rekursivität und Konstruktivität. In Heyting A (ed) Constructivity in mathematics. North-Holland Publishing Company, Amsterdam.

Sundholm G (2011) A garden of grounding trees. In: Cellucci C Grosholz E and Ippoliti E (eds) Logic and Knowledge. Newcastle, Cambridge Scholars Publishing, p 57-74

Thank you