What is Ordinary Mathematics?

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Structure of the talk

The Distinction: a Historical Characterisation I

- Set-theoretic mathematics includes those branches of mathematics that were created by the set-theoretic revolution.
  - E.g. the more abstract forms of point-set topology and functional analysis, as well as set theory itself.
- Ordinary mathematics includes those branches of mathematics that are historically independent of the development of set theory in the sense that they are *prior to* or *independent* of the introduction of abstract set-theoretic concepts.
  - E.g. real and complex analysis, geometry, countable algebra, number theory, and combinatorics.

(Simpson 2009, p. 1)

Our leading question referred to "ordinary mathematics". By ordinary mathematics we mean, roughly speaking, mainstream or non-set-theoretic mathematics, i.e. mathematics as it was before the abstract set theorists got hold of it (or perhaps: as it would have been if the abstract set theorists had never gotten hold of it). Thus ordinary mathematics includes number theory, geometry, calculus, differential equations, real and complex analysis, combinatorics, countable algebra, separable Banach spaces, computability theory, and the topology of complete separable metric spaces. It does not include abstract functional analysis, abstract set theory, or general topology.

(Simpson 1985, p. 461)

A Historical Characterisation II

Simpson uses this distinction to delimit the subject matter of reverse mathematics (RM), which focuses on the question:

> *Which set existence axioms are needed to prove the theorems of ordinary, non-set-theoretic mathematics?*

A historical characterisation seems to fail to get a grip on this distinction.

Many areas that we would include within ordinary mathematics (e.g. arithmetic combinatorics) developed after the birth of set theory, and presumably ordinary mathematics will keep developing.

- There are good reasons to think that the distinction between ordinary and set theoretic mathematics is a conceptual one.
- There are also good reasons to think that Simpson is intending to make a conceptual distinction here.

A Conceptual Characterisation: Expressibility

Simpson also characterises this distinction in terms of expressibility:

- The objects of ordinary mathematics are those which admit of finite or countable representations.
- While the objects of set-theoretic mathematics also include uncountable objects which cannot be countably represented.

For this reason, the reverse mathematical study of mathematical theorems is carried out in the context of the language of second order arithmetic.

Reverse Mathematics and Unification

If sound and substantial, this distinction would provide an interesting picture of mathematics as divided into two parts: ordinary mathematics on the one hand, set-theoretic mathematics on the other. Reverse mathematics, on this picture, would act as a *unifying framework* for ordinary mathematics.

Not only can ordinary mathematics be expressed in second order arithmetic, but working in a weak base theory, we can show how many different theorems of ordinary mathematics imply one another, despite coming from many different areas with different basic concepts.

- For example, some theorems of group theory are equivalent to theorems of functional analysis.
- Similar phenomena appear also in set-theoretic mathematics: e.g. the many statements equivalent to the axiom of choice and its weakenings, such as Tychonoff's theorem.

The Language of Second Order Arithmetic

- The language of second order arithmetic quantifies over two sorts of objects: natural numbers, and sets of natural numbers.
- It can directly represent finite and countable objects.
- Uncountable objects must be represented indirectly, by means of countable codes.

The Theory of Second Order Arithmetic

The full theory of second order arithmetic or Z_2 consists of:

- The *basic axioms* of PA^-;
- The second-order *induction axiom*;
- The *second order comprehension scheme* stating that every set of natural numbers definable by a formula in the language of second order arithmetic exists.

Subsystems of Second Order Arithmetic

$$\begin{array}{c}
\mathsf{Z}_2 = \Pi^1_\infty\text{-}\mathsf{CA}_0 \\
\Downarrow \\
\Pi^1_1\text{-}\mathsf{CA}_0 \\
\Downarrow \\
\mathsf{ATR}_0 \\
\Downarrow \\
\mathsf{ACA}_0 \\
\Downarrow \\
\mathsf{WKL}_0 \\
\Downarrow \\
\mathsf{RCA}_0
\end{array}$$

The Expressibility Thesis

The language of second order arithmetic permits quantification over countable sets, so there is a sense in which it is adequate to represent both countable objects such as countable algebraic structures, and uncountable but countably-codeable objects, such as the real numbers.

> *The* ***expressibility thesis*** *is the claim that ordinary mathematics is that part of mathematics which can be faithfully represented within second order arithmetic.*

In other words, the extension of the notion of ordinary mathematics consists in the body of mathematical statements that can be faithfully expressed in the language of second order arithmetic.

Counterexamples to the Expressibility Thesis I

However, there are theorems which are both expressible in the language of second order arithmetic, and yet intuitively appear to be set-theoretic in nature.

These include *axioms of definable determinacy*, which include the following two candidates:

- Borel determinacy. Every infinite two-player game of perfect information with a Borel payoff set is determined (that is, one player has a winning strategy).
- Projective determinacy. Every infinite two-player game of perfect information with a projective payoff set is determined (that is, one player has a winning strategy).

Definable Determinacy Axioms are Set-Theoretic Principles

Intuitively, Borel determinacy and projective determinacy are both essentially set-theoretic in nature.

- Borel determinacy is provable in ZFC (Martin 1975), but it requires the Axiom of Replacement (Friedman 1971). In fact, it requires uncountably many iterations of the powerset axiom.
- Projective determinacy is not provable in ZFC, but it is provable in ZFC plus certain relatively strong large cardinal axioms (Martin and Steel 1989).

Counterexamples to the Expressibility Thesis II

Projective determinacy and Borel determinacy can both be expressed schematically in second order arithmetic.

So by the expressibility thesis, they are theorems of ordinary mathematics.

However, they seem to have a clear set-theoretic character, and thus are part of set-theoretic mathematics.

- This undermines the expressibility thesis as a way of spelling out the distinction between ordinary and set-theoretic mathematics.

A Response

A supporter of this view could insist that their essential use of set-theoretic methods and concepts makes these counterexamples fall outside the scope of ordinary mathematics.

- However, it is difficult to see how such an argumentative strategy could be pursued without appealing to *ad hoc* modifications of the expressibility view.
- Moreover, such modifications appear circular, since in making them one presupposes the prior availability of a principled distinction between set-theoretic and non-set-theoretic mathematics.

In the absence of persuasive responses to counterexamples of this kind, the expressibility view does not constitute an adequate elucidation of the distinction between ordinary and set-theoretic mathematics.

An Alternative to Expressibility

The following quote from Mariagnese Giusto illustrates a somewhat different intuition about the distinction between ordinary and set-theoretic mathematics.

> *By "ordinary mathematics" we mean those parts of mathematics which do not essentially need uncountable ordinals and cardinals, which are essential in the so-called "set-theoretic mathematics". [...] The reason for this restriction is that the set existence axioms which are needed for "set-theoretic mathematics" are likely to be much stronger that those which are needed for "ordinary mathematics".*

(Giusto 2003, p. 64)

An Analogy with Arithmetic I

Counterexamples to the expressibility thesis like Borel determinacy are reminiscent of the situation in first-order arithmetic.

- The search for "natural" examples of incompleteness led to the discovery of statements like the strengthened finite Ramsey theorem and Goodstein's theorem.

  These are *expressible* in the language of first-order arithmetic, but *not provable* in the canonical first-order arithmetical theory of PA.

One view, analogous to our expressibility thesis, is that these statements are truths of arithmetic: despite being unprovable in PA, their fundamental subject matter is the natural numbers.

An Analogy with Arithmetic II

A contrasting position has been taken by Daniel Isaacson (Isaacson 1987, 1992), who argues that first-order PA is *sound and complete with respect to our notion of arithmetical truth*.

This has been labelled Isaacson's thesis.

On this view, proving any true statement which is expressible in $\mathcal{L}_{\mathrm{PA}}$ but independent of PA requires an appeal to concepts that go beyond those that are required in understanding PA (what Isaacson refers to as "hidden higher-order concepts").

The Epistemic Thesis I

We can state a thesis similar to Isaacson's, but for second order arithmetic.

> *If we are to give a proof for any true sentence of the language of second-order arithmetic L_2 which is independent of Z_2, then we will need to appeal to ideas that go beyond those that are required in understanding Z_2.*

This gives rise to the following alternative to the expressibility thesis.

> *The epistemic thesis is the claim that ordinary mathematics is that part of mathematics which can be proved within Z_2.*

The Epistemic Thesis II

The epistemic thesis thus holds, in a nutshell, that ordinary mathematical knowledge is the mathematical knowledge obtained on the basis of our grasp of the concepts expressed by the formal system of second order arithmetic Z_2.

The idea here is that certain methods of proof rely on our grasping certain *hidden higher-order concepts*. On this account, proofs of Borel determinacy essentially involve set-theoretic methods, and thus we must understand the theorem itself as set-theoretically entangled.

- The epistemic thesis can therefore accommodate the counterexamples to the expressibility thesis.
- The hope is that the epistemic thesis can do better justice to the distinction between ordinary and set-theoretic mathematics than the expressibility thesis.

Completeness

Isaacson argues that PA is *complete* for arithmetical truth: every true arithmetical statement can be proved in the formal system PA.

The epistemic thesis likewise holds that Z_2 is complete for ordinary mathematical truth: every true statement of ordinary mathematics can be proved in the formal system Z_2.

The technical literature seems to give us hope that this is true: statements of ordinary mathematics do seem to be provable in (generally quite weak subsystems of) Z_2.

However, it is less clear that Z_2 is *sound* for ordinary mathematical truth, in the sense that there are no statements φ such that $\mathsf{Z}_2 \vdash \varphi$, and yet φ is essentially set-theoretic in nature.

Soundness

While Z_2 cannot prove Borel determinacy, it can prove a number of other theorems from descriptive set theory.

- At the low end, the subsystem ATR_0 proves the axiom of determinacy for open sets, and Π^1_1-CA_0 proves the Cantor/Bendixson theorem.
- At the high end, Z_2 can prove quite a bit of determinacy (Montalbán and Shore 2012).

This might make one worry that the epistemic thesis as stated above has somehow overreached: ordinary mathematics, if understood in terms of proof-theoretic strength, perhaps only reaches as high as Π^1_1 comprehension or so.

However, this is a subtle issue since statements like the Cantor/Bendixson theorem or coanalytic uniformization could be considered part of analysis rather than being set-theoretic in any strong sense.

Conclusions

- The historical characterisation of the distinction between ordinary and set-theoretic mathematics is not a viable one.
- The expressibility thesis suffers from serious counterexamples, which the epistemic thesis accommodates.
- Determinacy axioms provable in Z_2 make the picture more muddy: to be more confident in the epistemic thesis, we need to establish a more precise bound on the limit of ordinary mathematics, in terms of proof-theoretic strength.
    - At the lower end, theorems provable in ACA_0 clearly seem to belong to ordinary mathematics.
    - The amount of determinacy provable in Z_2 suggests that set-theoretic concepts are essential to this system, and thus that the epistemic thesis may not be sound for ordinary mathematics.
    - But there is a big gap between ACA_0 and Z_2!
    - Tentative suggestion: Π^1_1 comprehension may be the bound we are looking for.

Thank you.

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