A note about the notion of $\exp_{10}(\log_{10}(\text{modulo } 1))(x)$

Concise observations of a former teacher of engineering students on the use of the slide rule

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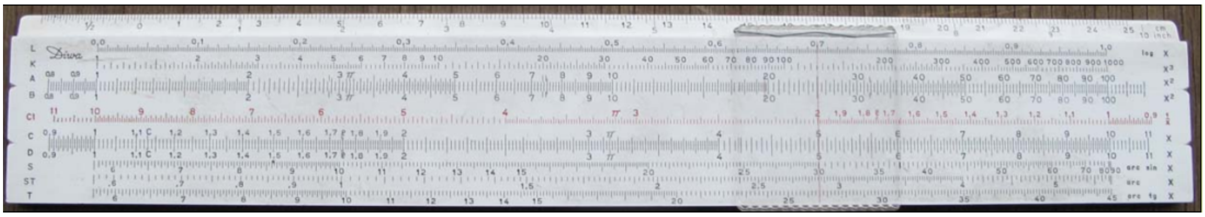
As is well known among those who are interested in Babylonian mathematics or mathematical astronomy, the Babylonian sexagesimal place value system was a floating-point system, that is, the notation contained no indication of absolute order of magnitude. A number 1 **2**5 might stand for any number $60^n\cdot(1\cdot60+25)$, where $n\in\mathbb{Z}$.[1] In the language of modern set theory and algebra, we may also say that it stands for the equivalence class of numbers whose logarithms (base 60) are congruent modulo 1 – the set $\exp_{60}(\log_{60}(\text{modulo } 1))(1\cdot60+25)$. If we introduce the notation "mod_m" for "modulo with respect to the multiplicative group of positive rational numbers", we may instead write $(1\cdot60+25)(\text{mod}_\text{m}\ 60)$ – apparently simpler, but presupposing the whole apparatus of modern group theory.[2]

Whatever the formulation, we are moving within second-order arithmetic.

Those who taught mathematics to secondary school or engineering students during the first half of the twentieth century (indeed, until the early 1970s) knew a very similar instrument, the slide rule:



The principle was that numbers from 1 to 10 were mapped onto a unit length (on the standard specimen I show, 25 cm) according to their decadic logarithm: in consequence, multiplication of numbers corresponded to the

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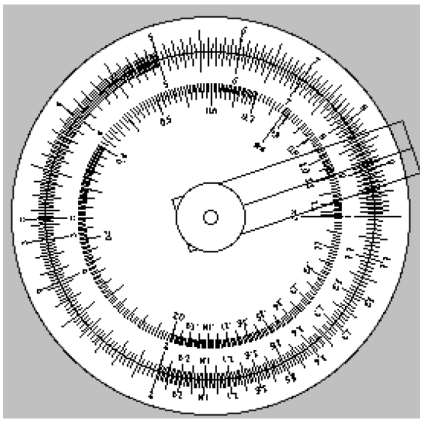
[1] In order to avoid zeroes not written by the Babylonians, I adopt the convention of writing multiples of 10 in bold digits, multiples of 1 in normal font. What is traditionally written 1 20 (or with some indication of order of magnitude) thus becomes 1 **2**.

[2] Since the Babylonian place value system was only used for numbers whose fractional part can be written as a finite sexagesimal fractions, it did not fill out this group; on the other hand, it was used to write "irregular numbers", that is, numbers whose reciprocal cannot be written in this way, and therefore goes beyond the multiplicative group of regular numbers. It can be described as the multiplicative semi-group $\mathbb{N}\cdot60^{\mathbb{Z}}$.

geometric addition of lengths, while division became geometric subtraction.[3] Evidently, products might well fall within the interval from 10 to 100, that is, on the prolongation of the unit interval, of which only the beginning (from 10 to 11) is indicated on the slide rule; similarly, quotients might fall in the interval from 0.1 to 1, of which only the end (0.9 to 1) is indicated. However, shifting the endpoint of the tongue, corresponding to a division respectively a multiplication by 10, would give the correct digits of the product or quotient within the interval from 1 to 10. Only the distinctions between "1.1" and "11" and between "0.9" and "9" rule out an interpretation where numbers are mapped onto the unit interval according to *the mantissa* of their decadic logarithm, in which case (e.g.) "5" would have represented all numbers $5{\cdot}10^n$, $n \in \mathbb{Z}$ – in other terms, the whole class 5 (mod_m 10).

A variant of the instrument was circular in shape:



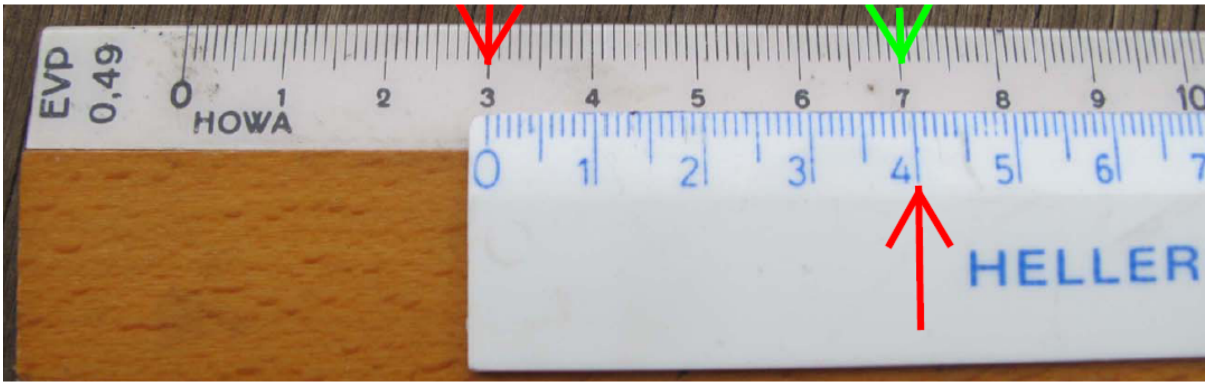
On this device, there can be no distinction between "1.1" and "11", and the mantissa-interpretation seems most adequate – "numbers with undetermined order of magnitude" or "equivalence classes (mod_m 10)".

Some of us who taught the use of the device had been trained on Bourbaki-style algebra, Riemann surfaces and multi-valued functions. If it had been our job to teach abstract group theory, the slide rule might have been a perfect illustration of the notions of subgroups and cosets. However, I have never heard about any such use, which may have been deemed too technical and too bound to applications to please Bourbaki's spirit; nor did I use it myself for this purpose during the four years where I was an

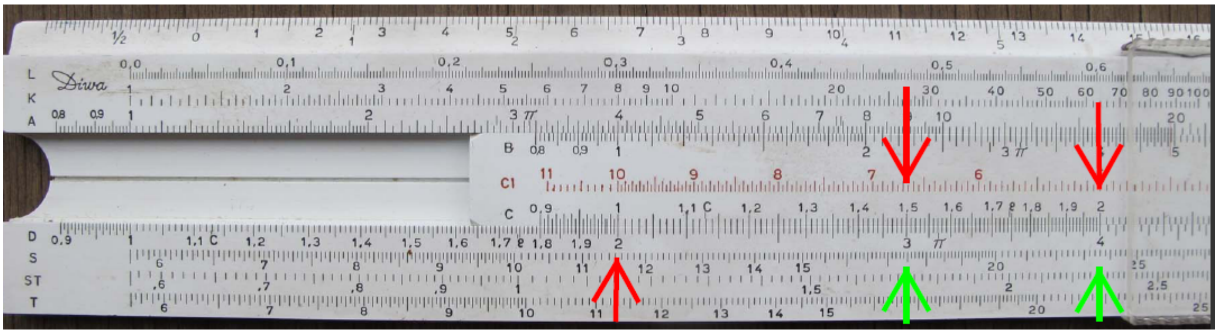
[3] I skip discussion of the various scales for powers and trigonometric functions.

university instructor of algebra. The slide rule was taught in view of computation.

How would one then teach the use of the slide rule? Certainly not in terms of equivalence classes in any formulation (including “numbers with undetermined order of magnitude”, which is just a way of “dividing out” the subgroup of powers of 10 and thus to produce the equivalence classes as cosets). If teaching students who already knew about logarithms, you might start by the additive analogue, showing how 3+4 can be found by means of two ordinary rulers:



and then go on with illustrative examples on the slide rule itself. But you might also go directly to these illustrations, as would anyhow be necessary in the teaching of students who did not know about logarithms – for instance



Here, you might point out the miracle that the operation on 2 and 2 does indeed give 4 (or something very close to it), while that on 2 and 1.5 yields 3. In the next step, it could be explained that “2” and “1.5” might also stand for 20 and 15, and that the result would therefore have to stand for one of the other possible interpretations of “3” – namely 300. But you would always stick to explanations that a reading “*n*” of the slide rule *might* stand for specific numbers differing by a factor 10^n, never that it stood for the whole class of such numbers. The difference is similar to the one between first- and second-order logic. Even introductory teaching of logic (as all logic until less than two centuries ago) starts by first-order-logic – for instance

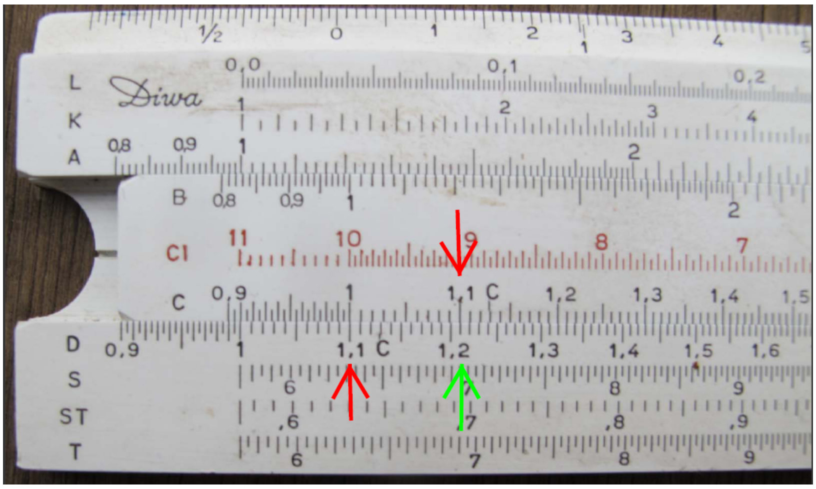
$$((p\Rightarrow q)\wedge\neg q)\Rightarrow\neg p\ ,$$

not by the trivial second-order embedding (S standing for the “space of propositions”)

$$\forall(p,q\in S)(((p\Rightarrow q)\wedge\neg q)\Rightarrow\neg p)\ .$$

Even in mathematics, as we know, genuine second-order formulations were rare (not quite absent, but obviously difficult to think of even for gifted mathematicians) before Weierstraß.

When calculating with the slide rule, students (or physicists, or engineers, etc.) had various ways to keep track of the intended order of magnitude. In simple cases it would just be known, as in this illustration of 11×11, determined as 1.1×1.1:



Nobody would need much reflection to pick the value 121 for the result – most calculators probably would not even take note that the factors are 1.1, not 11, and that the immediate outcome is 1.21, not 121. In complex calculations, one would for instance divide out powers of 10 and write them down separately, multiply (and divide) the factors and observe when new factors 10 or 10^{-1} resulted, and in the end combine the digits resulting on the slide rule with the power 10^n resulting for the counting of powers[4] – perhaps also reflecting on the verisimilitude of the outcome as an extra check when possible.[5]

In conclusion, *we* may explain the structure of the Babylonian place-value system as (a semigroup contained within) the quotient group of the group of positive rational numbers constituted by equivalence-classes (mod_m 60) – "numbers deprived of sexagesimal order of magnitude". We should not believe the Babylonian calculators less intelligent than we are – nor however that they were mathematically more bright than Euclid, who never got the idea to explain the doctrine of odd and even in terms of equivalence classes and groups, and who defined equality of ratios (*Elements* V, def. 5) almost as if it was done in first-order logic (current translations restitute a second-order formulation "any ... any", but Euclid's own words are much less clear). Second-order logic and its mathematical counterparts grew out not of a particular intelligence of our epoch but of a particular intellectual organization of university mathematics, deliberately separate from engineering applications (however much part of the outcome has then turned out to be highly relevant for advanced applications). Neither Babylonian scribe school students nor their teachers lived within a similar intellectual framework – *mutatis mutandis*, their world corresponded to that of engineering students and their teachers of the nineteenth and earlier twentieth century, when the teachers had been

4 *Within* a sequence of multiplicative operations on the slide rule, the calculator would thus not necessarily keep track of orders of magnitude but just think of the intermediate result as, e.g., 3.56 or 35.6.

5 Errors *were* of course made, not only by students but also by working engineers. Around the age of ten I read a book by Negley Farson; the title is no longer in my memory, but it may have been a Danish translation of *Going Fishing*. The only thing I remember from the book is a story about an engineer who because of a slide-rule error used 100 times as much dynamite as he should for a supposedly minor work on a river, with the result that the river was completely barred, and a salmon species spawning just then in this river and nowhere else was almost eradicated. I used the story as a warning to my engineering students when they believed that errors concerning only the place of the decimal point were less serious than other errors.

trained in the *cours-d'analyse*, not the Bourbaki tradition. Until sources teach us differently – and the sources are probably too meagre in this respect to tell us anything – we should therefore believe their thought and training in the use of floating-point calculation to have been more similar to that of almost-contemporary engineering students than to that of Bourbaki. To render 1 **2**5 as 1°25´ (or 1;25, in an alternative notation) gives more information than warranted by what is written on the clay tablet, but it is hardly a betrayal of what was on the mind of the scribe who wrote the signs – *if only* we keep in mind, as the scribe certainly did, that the signs taken in isolation might just as well mean 1´**2**5´ or 1`25, and that any interpretation will be legitimate as long as we stay within the context of the multiplicative group of rational numbers, though not within the corresponding defective additive-multiplicative semiring (to use words whose generality does not translate into anything a Babylonian calculator would recognize, but whose interpretation in the concrete case would not have told him anything new): once addition intervenes, at least the relative order of magnitude of addends has to be decided.

When making a transliteration of a cuneiform text we do not behave differently.[6] Today, nobody would produce a string of uninterpreted

[6] A brief and simplified explanation for non-Assyriologists: The Mesopotamian script was created in the later fourth millennium BCE, and was originally purely ideographic and pictographic, for which reason we cannot know to which language it corresponded – and in any case it was not used to render spoken language but for accounting. However, toward the mid-third millennium it started to be used in royal inscriptions and in literary texts (hymns, proverbs), that is, to render language – namely Sumerian – and the ideograms can now be understood as *logograms*, signs for particular words. Sometimes, simplification led to merger of original distinct signs, which could thus be read in several ways. In order to render language the scribes needed to indicate grammatical elements, and logograms were recycled for this purpose according to their *approximate* phonetic value – for instance, the same sign might be used for the syllables /bi/, /be/, /pi/ and /pe/). In the earlier second millennium, Sumerian had died except as a scholarly language known by scribes, and the logograms where now mostly meant to be pronounced in the current Akkadian language (of which Babylonian and Assyrian were the main dialects), even though some Sumerian values were taken over as loan-words. The phonetic values were conserved and new were sometimes added.

Moreover, from early times, some signs could serve as *determinatives*, which sign names: in as far as possible the transliteration has already decided upon the reading of each sign as a determinative, a logogram or a syllabogram (and *which* logogram respectively syllabogram). *We* know that *the scribe* knew about the ambiguities of the system, and also that our decision may sometimes be mistaken and is sometimes arbitrary – the Akkadian loanword *igûm* shows that the sign igi would on some occasions or in some environments be pronounced according to the Sumerian phonetic value, while occasional interlinear glosses make it clear that it would sometimes be thought of as a logogram for Akkadian *pani*, "in front of". In any case, the transliteration renders a possible and even likely interpretation of the thought of the scribe, although it may at times not hit the point exactly; however, to render it as if the author of the text had thought of *nothing but* sign names would be much further off the mark.

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were not meant to be pronounced but indicated the semantic class of the following sign, thus facilitating its interpretation but adding another ambiguity.

The sign lists that were used in scribal training indicate *sign names* for the signs, as a rule corresponding to a *possible* Sumerian logographic reading.