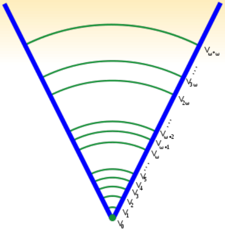

Interpreting the world set-theoretically?

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1. Indefinite Extensibility
2. Uzquiano's view
3. The unstable middle – Platonism or constructivism?

Inconsistent totalities

Both the set-theoretic paradoxes and the apparent arbitrariness of the thought of there being any upper bound on α in V_α have motivated the thought that the domain of sets is **indefinitely extensible**. We can perhaps see a hint of the idea in a letter from Cantor to Dedekind:

> *For a multiplicity can be such that the assumption that all of its elements 'are together' leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as 'one finished thing'. Such multiplicities I call absolutely infinite or inconsistent multiplicities.*

The proponent of indefinite extensibility emphasises the *finished* in 'one finished thing' (rather than the *one*). For her, problems arise when we treat the concept *set* (or *ordinal*, or. . .) as having a determinate extension.

Indefinite extensibility

Dummett:

> *[An] indefinitely extensible concept is one such that, if we can form a definite conception of a totality all of whose members fall under the concept, we can, by reference to that totality, characterize a larger totality all of whose members fall under it.*

Russell on Frege:

> *the contradictions result from the fact that . . . there are what we may call self-reproductive processes and classes. That is, there are some properties such that, given any class of terms all having such a property, we can always define a new term also having the property in question. Hence we can never collect all of the terms having the said property into a whole; because, whenever we hope we have them all, the collection which we have immediately proceeds to generate a new term also having the said property.*

Understanding indefinite extensibility

One way of cashing out indefinite extensibility is in terms of ontological indeterminacy.

Intuitionism

There simply is not a (classical) fact of the matter regarding the extension of some concepts. Therefore quantification over these concepts cannot be classical. Recognition of this phenomenon, therefore, forces mathematical revision. Thus Dummett himself.

Modalising

We should understand indefinite extensibility modally – we **can**/ **could** come up with a more expansive variant of any i.e. concept. Thus Linnebo's modal set theory. (nb: this is implemented in S4.3, which can be transformed Kripke-style into the superintuitionistic logic known as Gödel-Dummett logic).

Uzquiano's alternative

Linguistic reinterpretation account of indefinite extensibility:

> Keeping the domain fixed, it is always possible to extend the interpretation of 'is a set' and '$\in$' so that more entities/ pairs fall within their extensions.

- Developed formally in a fixed-domain model logic.
- We read $\ulcorner \Diamond\phi \urcorner$, 'It is possible to reintepret the language s.t. ϕ'.
- Some care is needed in order to avoid paradox.

Uzquiano's formulation of I.E.

Where $\alpha(x)$ reads 'x is available to form a set' and '$x \equiv xx$' reads 'x is the set of xx':

$$\forall xx(\forall x(x \prec xx \rightarrow \alpha(x)) \rightarrow \exists x \equiv xx) \qquad \text{(COLLECTION)}$$

Where the diamond is interpretational:

$$\forall x(\mathtt{Set}(x) \rightarrow \Diamond\alpha(x)) \qquad \text{(AVAILIBILITY}^{\Diamond}\text{)}$$

$$\forall x(\mathtt{Set}(x) \rightarrow \Box\mathtt{Set}(x)) \qquad (\Box\text{Set})$$

Sethood is interpretation relative

- There is no interpretation-independent fact of the matter whether some entity is a set.
- In particular we don't have: $\forall x(\Diamond \mathtt{Set}(x) \rightarrow \mathtt{Set}(x))$.

Uzquiano:

> *Perhaps we should think of a set as a mere node in a structure that satisfies certain formal conditions imposed by the axioms at the outset. The set-theoretic universe could perhaps be reduced to a domain of objects related by a formally appropriate relation that satisfies the relevant axioms. . .*

A minor metaphysical worry

Here is a principle that is widely accepted, at least implicitly:

NMC

No mathematical object is identical with any concretum.

It's not clear that Uzquiano's view delivers **NMC**. What is to stop me re-interpreting 'is a set' so that Julius Caesar is a set?:

- If the answer is 'something' then the account as it stands is incomplete, since this is not explained.
- If the answer is 'nothing' then that is a cost of the account. There are also Lewisian worries ('If Tiddles dies, will mathematics fall?').

A semantic worry

For Uzquiano, set-theoretic terms refer to entities. But this reference looks **arbitrary**.

The use constraint violated!

What in our *use* of set-theoretic language secures that a given set-theoretic term refers to *a* rather than to distinct *b*? It looks like we have use-transcendent meaning. But this surely violates a reasonable constraint on a theory of meaning, that meaning must be manifestable in use (and so communicable and learnable).

Structuralism

Note that Uzquiano talks about:

> *. . . a set as a mere node in a structure that satisfies certain formal conditions imposed by the axioms*

This looks like a version of *in rebus* structuralism.

- There is no more to being a set than satisfying '`Set`' in an interpretation of the language which yields a formally appropriate membership relation.
- It is the *structure*, not the objects, that matter.
- c.f. 'Mathematics is the science of structure'.

Not enough objects?

A familiar objection to *in rebus* structuralism is that there may not, by the structuralist's ontological lights, be enough objects to instantiate the structures described by mathematical theories:

> Consider, for example, an *in rebus* structuralist who only accepts physical entities. In order to account for the structure $(\mathbb{R}, <)$ we need there to be a physical continuum. This makes mathematical truth hostage to physical facts. But this looks wrong.

Uzquiano admits higher-order quantification. So effectively we're concerned with ZFC2. The smallest model of this has $|\kappa|$, κ inaccessible (given standard semantics).

No circularity!

Why believe there are inaccessibly many objects?

One good reason

We believe set theory!

But this is fatally circular in context. We need to be justified in accepting a sufficiently sized ontology for reasons independent of set-theory in order to motivate an understanding of indefinite extensibility in terms of linguistic re-interpretation.

Where are we at?

Uzquiano's view as *moderate realism*

- On Uzquiano's view, mathematical reference is to an external, independent reality.
- Ontology is determinate.
- But nonetheless *our practices* determine the extension of the concept *set*.

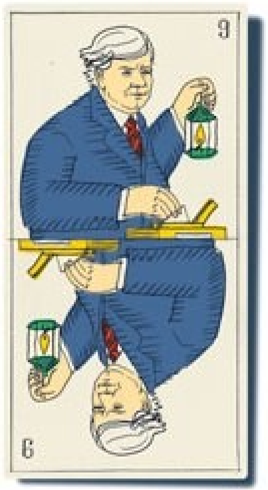
We can think of the view as treading a middle path between **platonism** and **constructivism**.

Are we forced to extremes?

- The worry about guaranteeing ontology might tempt us to think we are forced to an extreme.
- The challenge is to *explain*, in a justificatory fashion, our confidence that there are sufficiently many entities.
- The platonist does this with reference to an independent realm of mathematical objects.
- The constructivist does this with reference to our construction of the requisite ontology.

Worries for platonism

- How is reference to abstracta secured?
- How does set-theoretic practice track truth about these abstracta?
- How is indefinite extensibility to be understood platonistically? Do we have some form of ontic vagueness?
- How, in particular, do we explain away the constructivist ‘feel’ of i.e.?
- Set-theoretic pluralism.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Worries for constructivism

- The phenomenology of mathematical discovery.
- Doesn't constructivism force logic revision upon us?
- Worries about the applicability of mathematics.
- Worries about consistency.

My own preference

Quasi-realism

I would like to develop a version of **quasi-realism** about set-theory:

- Our set-theoretic practice has explanatory priority.
- This doesn't mean there is any sense in which set-theory is untrue, or sets are unreal. It is this practice that places us in a position to say that there are sets.
- The (implicit) second-order and classical nature of set-theoretic practice delivers us quasi-categoricity and classically behaved truth.
- I think supervaluation over ω-universes is a reasonable approach to set-theoretic pluralism.


Merci
beaucoup