PROJET DE THESE

The aim of this project is to explore the research about the determination of the irrationality of pi in XVIIth century mathematics. The project will be developed from both philosophical and historical viewpoint.

HISTORICAL VIEWPOINT

The problem pertains from one hand, to the history of analysis (particularly with respect to the study of series and continuous fractions) and to the history of geometry, from the other: the determination of the value of pi was actually connected to the efforts for determining the area of the circle, dating back to Greek mathematics.

By means of the XVIIth century contributions, ancient tradition exerts a strong influence on the development of mathematics during the next century. To examine the limits and the pregnancy of this influence on XVIIIth century mathematics will be one of the main aims of this project. The problem of quadrature of the circle will be taken as a case study.

The present work will start directly from the contributions on the quadrature done by Leibniz[1] and Gregory[2], but also from the polemic between the latter and Huyghens, with respect to the impossibility to perform the quadrature via analytical methods[3], and the reactions issued within the *Royal Academy.* Secondly, the present work will concentrate on the most advanced research of those mathematicians who were used to frequent these three cultural and scientific centers of the time: Berlin, Paris, and St Petersburgh.

The following mathematicians will be studied in detail: Johann Heinrich Lambert (1728-1777), who published a proof of the irrationality of pi in 1768[4]; Adrien Marie Legendre

1 V. **[12]**.

2 Cfr. **[6]**, **[7]**

3 ***[27]***.

4 **[11]**, **[12]**. In these articles, Lambert shows a method for developing continuous fractions so as to avoid various complex calculations (cfr. **[3]**). this method was supposed to be applicable to the study of trigonometrical series and of some quadratic and cube roots. The link between these works and the research on the quadrature of the circle is evident in **[12]**, in which the infinite series of tan *v = a/b* (*sin v/cosv, where v express the commnesurable ratio between the radius and an arc of circle*) is transformed into a continuous fraction. At this point, Lambert applies Euclid's algorithm to determine the quotients Qi et the residues Ri. Clearly, the algorithm will stop when we reach a certain residue Rn = 0. Though, as Lambert shows that the greatest common divisor to all the Ri (which is indeed the greatest common divisor to A and B), is less than any assignable quantity, A and B cannot have a common divisor. Tan v = a/b is indeed irrational if v is an aliquot part of the radius. By converse, if tan v = a/b is rational, then v is not an aliquot part of the radius. Finally, if we put v = pi/4, we will have tan v = 1. Consequently, pi/4 and pi are irrational quantities.

(1752-1833), who amended on some points Lambert's demonstration[5], Jean le Rond D'alembert (1717-1783), who discussed the problem of the quadrature in his *Encyclopédie,* Alexis Claude Clairaut (1713-1765) who published the *Eléments de Géométrie* (p. 1741)[6], and Leonhard Euler (1707 - 1783), the founder of the investigations upon series and continuous fractions. We would like to examine in detail the work of John Playfair (1748 -1819)[7] in order to throw light upon the state of affairs in the scientific establishment in XVIIIth Great Britain. .

This research would like to resume the historical model developed by Henk Bos[8], concerning the study of geometrical constructions. This model envisages a "changing entity", called subject, within a larger domain ("context"), in which the subject undergoes some transformations following a "principal dynamic".

Hypothetically, we will choose as our main subject the problem of the determination of the ratio between the radius and the circumference of the circle. This choice refers on one hand to the context of geometry and to the classical problem of the quadrature. On the other, it represents a way to be faithful to history: in his essay, Lambert refers directly to this question, approaching it via those analytical methods developed by Euler[9].

This choice meets one of my broader purposes: establishing a link between geometrical and analytical ways of dealing with the problem, in order to explore in which ways the evolution of analysis (according to Carl Boyer's model) has influenced the evolution of my subject[10].

Context will be dealt with through a detailed conceptual and technical analysis, inspired on one hand by Montucla's model of history of mathematics[11], and by Klein[12] and Hobson[13] on the other. We should be able then to show which technical improvements were introduced during XVIIIth century: the study of series and continuous fractions.

Meanwhile, these models could be enriched and should be enriched, too. The mere presentation of a group of facts, the detailed and exhaustive examination of documents, and the exact reference to original manuscripts stands as a necessary condition of the historical work.

Nevertheless, other elements are likely to demand consideration from the historiographic side. First of all, an historical examination of methods of proof, as well as an examination of the different ways through which mathematical reasoning has deployed: the role of inconsistencies in Lambert's and Legendre's proofs, and the role of modifications they have undergone during the centuries could

5 **[13]**.
6 **[1]**.
7 **[15]**.
8 Dans ***[2]***,
9 **[12]**.
10 ***[13]***.
11 ***[20]***.
12 ***[13].***
13 ***[10]***.

be read with the aid of their own proper standards of rationality.

Secondly, Klein and Hobson's works show a concise though useful framework, in order to fix a chronology in the history of the problem, even if they risk to engender a too strict subdivision into intervals whose chronological limits are often arbitrary.

Hobson's scheme will be taken as a working hypothesis to characterize a peculiar "subject dynamics": the application of analytical research to the study of a geometrical problem can be seen as the accomplishment of a tendency already begun in XVIIth century, then pursued by Lambert and Legendre.

PHILOSOPHICAL VIEWPOINT

From the methodological perspective of the present work, the historical analysis should be supported by a philosophical analysis of the given fragment of mathematics. Our empirical model has been partly represented by Kenneth Manders[14]: the main role of mathematics consists in the constitution of conceptual settings which I will call from now on "theories", considered as a set of stipulations, justifications for these stipulations, and propositions accompanying proper mathematical activity. The historical example which will be dealt with will allow us to study, according to the model evoked by Manders, a phenomenon of conceptual resetting (this word will be taken to mean any change allowing to organize and clarify the content of a peculiar fragment of mathematics, or of a non mathematical domain).

The application of new methods for the determination of the ratio between radius and circumference of a circle, as those recurring to continuous fractions, stands as a non trivial accomplishment[15], open to criticism, whose epistemological saliency must be rightly considered, particularly with regard to the fundamental question: "what element of a theory undergo a change under a conceptual reshaping?".

Any modern treatise of algebra[16] shows that the transcendence of pi is a necessary and sufficient condition to establish the impossibility of the quadrature of the circle: according to the analysis of documents, though, the irrationality of pi was was likely to be accepted, among XVIIIth century mathematicians, as a crucial result with respect to a decisive answer to problem of quadrature[17]. The example of the irrationality of π shows a non linear development: Lambert's, Euler's and Legendre's works claim the impossibility of the quadrature, without a rigorous proof of

14 ***[19]***.
15 ***[20]***.
16 ***[4]***; ***[28]***
17 **[2].**

it. This divergence between epistemic and logical significance of a proof will be deeply explored.

Various proofs of the irrationality of pi were edited after Lambert's and Lagrange's. A comparison between Lambert's, Hermite's and Laczkovich's proofs[18] could give some elements in order to question on the historical evolution of such notions as: “rigor”, “economy”, “acceptability” and explicativity (?) of a mathematical proof.

2b. On the transcendence of π

As a last point with respect to the problem at stake, I would like to explore the questioning aroused by the introduction of the concept of “transcendental number”, and its use during XVIII[th].

This notion is characterized precisely with Lambert[19]. It is anyway interesting to remark that Lambert conjectures the transcendence of pi already in **[12]**, whereas he proves just its irrationality (for a demonstration of the transcendence we should wait Lindemann, in 1882 in a different context). Lambert's example demands, according to us, the need of supporting the historical analysis by an epistemological analysis of the notions of "conjecture" and proof in mathematics[20]. We may then be able to answer to the following questions: what is the epistemological value of this conjecture? What is its historical value? As well as to more general questioning, regarding the status of conjecture in mathematics as a whole.

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18 ***[16]***; ***[17]***
19 **[12]**.
20 ***[17]***.

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